Problems to Week 9 Tutorial — MACM 101 (Spring 2025)

1. Prove each of the following for all $n \geq 1$ using the principle of mathematical induction.

   (a)
   $$1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

   (b)
   $$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$

   (c)
   $$\sum_{i=1}^{n} i(i!) = (n+1)! - 1$$

2. *(Only for those familiar with complex numbers)* Prove DeMoivre's theorem
   $$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

3. Prove that for all natural numbers n if $n > 3$ then $2^n < n!$.

4. A jigsaw puzzle is put together by successively joining pieces that fit together into blocks. A move is made each time a piece is added to a block, or when two blocks are joined. Use strong induction to prove that no matter how the moves are carried out, exactly $n-1$ moves are required to assemble a puzzle with n pieces.

5. Let $P(n)$ be the statement that a postage of n cents can be formed using just 3-cent stamps and 5-cent stamps.

   (a) Show that the statements $P(8)$, $P(9)$, and $P(10)$ are true, completing the basis step of the proof.

   (b) What is inductive hypothesis of the proof?

   (c) What do you need to prove in the inductive step?

   (d) Complete the inductive step for $k \geq 10$.